On Differentials of Functions in only certain cases *

Leonhard Euler

§337 If y was any function of x and this variable quantity is augmented by the increment ω that x goes over into $x+\omega$, then the function will take this value

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.}$$

and hence will receive this increment

$$\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.},$$

as we showed above [§ 48]. Therefore, if it is $\omega = dx$ such that x grows by the amount of its differential dx, then the function y will receive the increment

$$=dy+\frac{1}{2}ddy+\frac{1}{6}d^3y+\frac{1}{24}d^4y+\text{etc.},$$

which will be the true differential of y. But since any arbitrary term of this series has an infinite ratio to the following, with respect to the first all vanish such that taken dy in usual manner yields the true differential of y. In similar manner the true second, third, fourth etc. differentials of y will behave as follows

*Original title: "De Differentialibus Functionum in certis tantum Casibus", first published as part of the book „*Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755", reprinted in in „*Opera Omnia*: Series 1, Volume 10, pp. 542 - 563 ", Eneström-Number E212, translated by: Alexander Aycock for the „Euler-Kreis Mainz"

$$
\begin{aligned}
dd.y &= ddy + \frac{3}{3}\, d^3y + \frac{7}{3\cdot 4}d^4y + \frac{15}{3\cdot 4\cdot 5}d^5y + \frac{31}{3\cdot 4\cdot 5\cdot 6}\, d^6y \; + \text{etc.}\\
d^3.y &= d^3y + \frac{6}{4}\, d^4y + \frac{25}{4\cdot 5}d^5y + \frac{90}{4\cdot 5\cdot 6}d^6y + \frac{301}{4\cdot 5\cdot 6\cdot 7}\, d^7y \; + \text{etc.}\\
d^4.y &= d^4y + \frac{10}{5}d^5y + \frac{65}{5\cdot 6}d^6y + \frac{350}{5\cdot 6\cdot 7}d^7y + \frac{1701}{5\cdot 6\cdot 7\cdot 8}\, d^8y \; + \text{etc.}\\
d^5.y &= d^5y + \frac{15}{6}d^6y + \frac{140}{6\cdot 7}d^7y + \frac{1050}{6\cdot 7\cdot 8}d^8y + \frac{6951}{6\cdot 7\cdot 8\cdot 9}\, d^9y \; + \text{etc.}\\
d^6.y &= d^6y + \frac{21}{7}d^7y + \frac{266}{7\cdot 8}d^8y + \frac{2646}{7\cdot 8\cdot 9}d^9y + \frac{22827}{7\cdot 8\cdot 9\cdot 10}d^{10}y + \text{etc.}\\
&\qquad\qquad \text{etc.}
\end{aligned}
$$

which follow from § 56, if instead of ω one puts dx. Therefore, these differentials of y will be complete, in which not even the terms which with respect to the first vanish are neglected. But these single terms are found, if the function y is continuously differentiated by putting dx constant. So having put $y = ax - xx$ because of

$$dy = adx - 2xdx \quad \text{and} \quad ddy = -2dx^2,$$

the complete differentials of y will be

$$dy = adx - 2xdx - dx^2, \quad ddy = -2dx^2;$$

the following are all zero.

§338 Although in general in these expressions of the differentials the following terms with respect to the first are considered to be nothing, nevertheless in special cases in which the first term itself vanishes this assumption is not valid and the second term cannot longer be neglected. So, even though in the preceding example the differential of the formula $y = ax - xx$ in general is $= (a - 2x)dx$ having neglected the term $-dx^2$, which certainly is infinitely smaller than the first $(a - 2x)dx$, here nevertheless this condition is manifestly understood, that the first term does not vanish per se. If therefore the differential of $y = ax - xx$ is in question in the case in which $x = \frac{1}{2}a$, then one has to say that it is $= -dx^2$; if the variable x grows by the differential dx, then the decrement of the function y in the case $x = \frac{1}{2}a$ will be dx^2. But, having excepted this single case, the differential of the function y will always be $= (a-2x)dx$; for, if it is not $x = \frac{1}{2}a$, the second term $-dx^2$ with respect to the first is always correctly neglected. And the negligence of the term dx^2 even in the case $x = \frac{1}{2}a$ cannot induce an error; for, the first differentials are used to be compared to each other; hence, because $dy = -dx^2$ in the case $x = \frac{1}{2}a$ vanishes in comparison to the first differentials dx, it does not matter, whether in this case we have $dy = 0$ or $dy = -dx^2$.

§339 While y denotes any function of x, let, having taken the continued differentials, be

$$dy = pdx, \quad dp = qdx, \quad dq = rdx, \quad dr = sdx \quad \text{etc.}$$

Hence, the complete differentials in which nothing is neglected of y will be

$$\begin{aligned}
&d.y \ = pdx \ + \frac{1}{2}q\ dx^2 + \frac{1}{6}r\ \ dx^3 + \frac{1}{24}sdx^4 + \frac{1}{120}tdx^5 + \text{etc.}\\
&d^2.y = qdx^2 + r \quad dx^3 + \frac{7}{12}sdx^4 + \frac{1}{4}t\ \ dx^5 + \text{etc.}\\
&d^3.y = rdx^3 + \frac{3}{2}s\ dx^4 + \frac{5}{4}t\ \ dx^5 + \text{etc.}\\
&d^4.y = sdx^4 + 2t\ \ dx^5 + \text{etc.}\\
&d^5.y = tdx^5\ + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

Therefore, if the first terms of these expressions do not vanish, they alone will exhibit the differentials of y; but, if in a certain case the first term becomes $= 0$, then the following will express the differential in question. And if also the second term vanishes, then the third term will yield the value of the differential in question; but if even this term vanishes, the fourth, and so forth. Hence, it is understood that the first differential of no function of x ever vanishes completely; for, even though it is $p = 0$ in which case sy is usually considered to vanish, this differential will then be expressed by means of a higher power if dx, as either by means of $\frac{1}{2}qdx^2$ or, if also $q = 0$, by means of $\frac{1}{6}rdx^3$, and so forth.

§340 But although in these cases the differential of y with respect to higher first differentials to which it is compared is correctly neglected and considered as zero, it is nevertheless often helpful to know also its true expression. For, from the complete form of the differential it is immediately seen in which cases the given function becomes maximal or minimal. For, if it was

$$d.y = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.},$$

that y obtains the maximum or minimum value it is necessary that $p = 0$; therefore, in this case it will be $dy = \frac{1}{2}qdx^2$ and the function y, if instead of x one puts $x \pm dx$, goes over into $y + \frac{1}{2}qdx^2$ and will therefore be minimal, if q has a positive value, but maximal, if q has a negative value. But if, at the same time $q = 0$, it will be $dy = \frac{1}{6}rdx^3$ and the function by putting $x \pm dx$ instead of x will go over into $y \pm \frac{1}{6}rdx^3$ and in this case neither a maximum nor a minimum arises; but if also $r = 0$, then having put $x \pm dx$ instead of x the function y will become $= y + \frac{1}{24}sdx^4$, which exhibits a maximum, if s was a negative quantity, a minimum on the other hand, if s is a positive quantity. Other occasions in which the complete expression of the differentials have a use will occur below.

§341 Let us put that p vanishes in the case $x = a$ what happens, if it was $p = (x-a)P$. But such a value arises, if it was

$$y = (x-a)^2P + C$$

while C denotes any constant quantity. For, because it is

$$pdx = (x-a)^2dP + 2(x-a)Pdx,$$

it will certainly be $p = 0$ having put $x = a$. Therefore, then because of

$$dpdx = qdx^2 = (x-a)^2ddP + 4(x-a)dPdx + 2Pdx^2$$

having put $x = a$ it will be $qdx^2 = 2Pdx^2$ and the complete differential in this case $x = a$ will be

$$dy = Pdx^2,$$

if not by accident also P vanishes for $x = a$ which cases I will contemplate later.

But the present case can be exhibited more general this way. Let

$$z = (x-a)^2P + C$$

and let y be any function of z such that $dy = Zdz$ while Z denotes any function of $z = (x-a)^2P + C$. Therefore, it will be

$$dz = (x-a)^2dP + 2(x-a)Pdx \quad \text{and} \quad pdx = Z(x-a)^2dP + 2Z(x-a)Pdx,$$

which term becomes $= 0$, if $x = a$; and in the same case having neglected the terms which contain the factor $x - a$ it will be $qdx^2 = 2PZdx^2$ and hence in the case $x = a$ it will be $dy = PZdx^2$, after in PZ is was put a for x everywhere. Hence, if y was any function of $z = (x-a)^2P + C$ such that it is $dy = Zdz$, in the case $x = a$ the differential will be

$$dy = PZdx^2.$$

Therefore, this function y becomes maximal in the case $x = a$, if in the same case PZ was a negative quantity, minimal on the other hand, if PZ was a positive quantity.

§342 If it was $p = (x-a)^2P$, in the case $x = a$ also q vanishes; but such an expression arises for p, if it was

$$y = (x-a)^3P + C.$$

Therefore, it will be

$$pdx = (x-a)^2dP + 3(x-a)Pdx,$$
$$qdx^2 = (x-a)^3ddp + 6(x-a)^2dPdx + 6(x-a)Pdx^2,$$

of which both sides vanish in the case $x = a$; but the following will be

$$rdx^3 = (x-a)^3d^3P + 9(x-a)^2ddpdx + 18(x-a)dPdx^2 + 6Pdx^3 = 6Pdx^3$$

having put $x = a$. Hence, because p and q vanish in the case $x = a$, it will be

$$dy = \frac{1}{6}rdx^3 = Pdx^3.$$

In similar manner, if one puts

$$z = (x-a)^3P + C$$

and y was any function of z such that it is $dy = Zdz$, because of

$$dz = (x-a)^3 dP + 3(x-a)^2 P dx$$

it will also be $p = 0$ and $q = 0$ and it will be $rdx^3 = 6PZdx^3$; hence, in the case $x = a$ it will be

$$dy = PZdx^3.$$

Hence, this function y, even though in the case $x = a$ it is $p = 0$, will nevertheless obtain neither a maximum nor minimum value.

§343 These differentials can be found more easily from the nature of the differentials itself. For, since the differential of y arises, if y is subtracted from the closest following state which arises, if instead of x one puts $x + dx$, let us put in the first case, in which it was

$$y = (x-a)^2 P + C,$$

$x + dx$ instead of x and it will be

$$y^{\mathrm{I}} = (x - a + dx)^2 P^{\mathrm{I}} + C,$$

whence it will be

$$dy = (x - a + dx)^2 P^{\mathrm{I}} - (x-a)P.$$

Therefore, in the case in which it is $x = a$ it will be $dy = P^{\mathrm{I}} dx^2$, and because P^{I} and P have the ratio of equality, it will be

$$dy = Pdx^2.$$

In similar manner, if it was

$$z = (x-a)^2 P + C,$$

it will be $dz = Pdx^2$; hence, if y is any function of z such that $dy = Zdz$, it will be

$$dy = PZdx^2$$

in the case in which one puts $x = a$.

Further, if it is

$$z = (x-a)^3 P + C,$$

it will be $z^{\mathrm{I}} = (x - a + dx)^3 P^{\mathrm{I}} + C$ and therefore in the case $x = a$ it will be

$$z^{\mathrm{I}} - z = dz = Pdx^3.$$

Hence, if y was any function of z and $dy = Zdz$, the differential in the case $x = a$ will also be

$$dy = PZdx^3,$$

if in the functions P and Z instead of x one substitutes a everywhere. Since in this case it is $z = C$ and Z is a function of z, Z will become a constant quantity, such a function of C, of course, as it was one of z before.

§344 Therefore, if it was in general

$$y = (x-a)^n P + C,$$

since it is

$$y^{\mathrm{I}} = (x - a + dx)^n P^{\mathrm{I}} + C,$$

in the case $x = a$ it will be

$$dx = Pdx^n;$$

whence, if it was $n > 1$, this differential will vanish compared to the higher first differentials which are homogeneous to dx. Therefore, from the preceding it is manifest that the function y becomes maximal or minimal in the case $x = a$, if n was an even number; for, then, if having put $x = a$ P becomes a positive quantity, y will be a minimum; but if P was a negative quantity, y will become a maximum. And this way the nature of maxima and minima is found a lot easier than by using the method explained above, since it is not necessary to go to higher differentials. If it is

$$z = (x-a)^n P + C$$

and y was any function of z that $dy = Zdz$, in the case $x = a$ the differential will be

$$dy = PZdx^n.$$

But it is to be noted that here n it to be taken positively or greater than 0; for, if n was a negative number, then for $x = 0$ $(x-a)^n$ would not vanish, as we assumed, but would even become infinitely large.

§345 Now, we saw that this way the differential can be found a lot more convenient than by means of the series by means of which we expressed the complete differential before; for, if n was an integer number, so many terms of that series have to be considered as n contains unities. But if n was a fractional number, then this series will not even ever exhibit the true differential. For, let us put that it is

$$y = (x-a)^{\frac{3}{2}} + a\sqrt{a};$$

if we consider the series

$$dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}dx^4 + \text{etc.},$$

it will be

$$p = \frac{3}{2}\sqrt{x-a}, \quad q = \frac{3}{4\sqrt{x-a}}, \quad r = \frac{-3}{8(x-a)\sqrt{x-a}},$$
$$s = \frac{9}{16(x-a)^2\sqrt{x-a}} \quad \text{etc.}$$

Hence, if one puts $x = a$, it will be $p = 0$, but all following terms q, r, s etc. will become infinite; hence, the value of the differential dy cannot be defined in this case at all. But the method deduced from the nature of differentials leaves no doubt. For, because it is $y = (x-a)^{\frac{3}{2}} + a\sqrt{a}$, having put $x + dx$ instead of x it will be $y^{\mathrm{I}} = (x-a+dx)^{\frac{3}{2}} + a\sqrt{a}$ and, if one puts $x = a$, it will be $dy = dx\sqrt{dx}$. Therefore, this differential vanishes with respect to dx; but the second differentials on the other hand homogeneous to dx^2 will vanish with respect to the latter.

§346 Let us expand these cases in which the exponent n is a fractional number a little more accurately and let be

$$y = P\sqrt{x-a} + C;$$

because of $y^{\mathrm{I}} = P^{\mathrm{I}}\sqrt{x-a+dx}+C$ it will be

$$dx = P\sqrt{dx}$$

in the case $x=a$; hence, this differential to dx and to the differentials homogeneous to dx will have an infinite ratio. Hence, it is also plain, what is to be said about the nature of maxima and minima in this case. For, because having put $a+dx$ instead of x y goes over into

$$C + P\sqrt{dx},$$

because of the ambiguous $\sqrt{dx}$ the function y will obtain two values; the one greater than C, which it receives for $x=a$, the other smaller; hence, in the case $x=a$ it will become neither a maximum nor a minimum. Furthermore, if dx is taken negatively, then the value of y will even become imaginary. The same is to said, if $z = P\sqrt{x-a}+C$ and y is any function of z that it is $dy = Zdz$; for, then it will be $dy = PZ\sqrt{dx}$ in the case $x=a$.

§347 If this function was propounded

$$y = (x-a)^{\frac{m}{n}}P + C,$$

whose differential is in question in the case $x=a$, it will be, as concluded from the preceding,

$$dy = Pdx^{\frac{m}{n}}.$$

Therefore, if it was $m>n$, this differential will vanish with respect to dx; but if it is $m<n$, the ratio $\frac{dy}{dx}$ will be infinitely large. Furthermore, if n is an even number, the differential will have two values, the one positive, the other negative; and so the function y which in the case $x=a$ becomes $=C$, if one puts $x=a+dx$, will have two values, the one greater than C, the other smaller; but if one would put $x=a-dx$, then y would even become imaginary; hence, in this case y would become neither a maximum nor a minimum. Now, let us put that the denominator n is an odd number; the numerator m will either be even or odd. Let m be an even number at first; since dy retains the same value whether dx is taken positively or negatively, it is perspicuous that the function y in the case $x=a$ becomes either maximal or minimal depending on whether in this case P was a negative or positive quantity. But if both numbers m and n were odd, the differential dy will go

over into its negative having put dx negative; and therefore, in this case the function y will be neither a maximum nor a minimum, if one puts $x = a$.

§348 If the function y consists of several terms of this kind, of which the single ones are divisible by $x - a$ such that it is

$$y = (x-a)^m P + (x-a)^n Q + C,$$

then its differential in the case $x = a$ will be

$$dy = Pdx^m + Qdx^n;$$

in this expression, if it was $n > m$, the second term vanishes in comparison to the first such that only $dy = Pdx^m$ arises. But if n was a fraction having an even denominator, then, even though Qdx^n vanishes with respect to Pdx^m, it can nevertheless not be completely neglected. For, from this it is clear, if dx is taken negatively, that the value of dy becomes imaginary, as from the first term Pdx^m alone is not plain. Therefore, since, if n is a fraction having an even denominator, dx cannot be taken negatively, but if it is taken positively, the term Qdx^n yields two values, the function $y = (x-a)^m P + (x-a)^n Q + C$ which in the case $x = a$ becomes $= C$, if $x = a + dx$, will be

$$y = C + Pdx^m \pm Qdx^n;$$

since both of these values is either greater or smaller than C, depending on whether P was a positive or a negative quantity, the function y in the case $x = a$ will be either a minimum or a maximum of the second kind [§278].

§349 Therefore, in the cases the true differentials of functions cannot be found by means of the usual rules for differentiation; these only hold, if the differential of the function is homogeneous to dx. But if in a singular case the differential of the function is expressed by means if its power dx^n, then the rule yields 0 for this differential, if n was a number greater than unity; but it on the other hands exhibits an infinitely larger differential, if n is an exponent smaller than unity. So if the differential of $y = \sqrt{a-x}$ is in question in the case $x = a$, since it is $dy = -\frac{dx}{\sqrt{a-x}}$, having put $x = a$ it arise $dy = -\frac{dx}{0}$. And if we wanted to derive the following differentials, all of them because of the denominators $= 0$ grow to infinity in the same way such that nothing can be concluded from there. But we saw that in this case it is $dy = \sqrt{-dx}$ and hence imaginary. But if instead of x one puts $x - dx$, it will be $dy = \sqrt{dx}$ and hence it will be infinitely larger than dx such that dx vanishes with respect to dy. Hence, the usual rule even in this case does not cause any errors, since it exhibits the infinite value of dy.

§350 Therefore, one has to go away from the usual rule, if in the series

$$pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.},$$

by means of which the complete differential of the function y is expressed, the first term p either becomes $= 0$ or infinite, and in this case the differential must be derived from the first principles. Therefore, if the differential corresponding to a given value, for which the letter p becomes either infinitely small or infinitely large, of x of the function y is in question, one has to go back to the first principles of differentiation. In all remaining cases in which neither $p = 0$ nor $p = \infty$ the usual rule will yield the true values of the the differential. Nevertheless the case mentioned before (§ 348) is not to be neglected, if the function y contains a term of the form $(x-a)^n Q$ while n is fraction having an even denominator; for, even though one has lower differentials than Qdx^n, with respect to which this one vanishes, nevertheless, since Qdx^n, if dx is negative, becomes imaginary, this term Qdx^n transforms all remaining ones, with respect to which it vanishes, also into imaginary ones; this circumstance is mainly to be considered in the case of curves. Therefore, I will explain some particular cases in which the true differential is not indicated by the common rule in the added examples.

EXAMPLE 1

Let the differential of the function $y = a + x - \sqrt{xx + ax - x\sqrt{2ax - xx}}$ be in question in the case in which one puts $x = a$.

It is plain that the differential of this function $x = a$ is not found by means of the usual rule; for, it is

$$dy = dx + \frac{-xdx - \frac{1}{2}adx + \frac{1}{2}dx\sqrt{2ax - xx} + (axdx - xxdx) : \sqrt{2ax - xx}}{\sqrt{xx + ax - x\sqrt{2ax - xx}}};$$

for, having put $x = a$ it will be $dy = dx - \frac{adx}{a} =$. Therefore, let us start from the principles of differentiation and at first having put $x + dx$ instead of x it will be

$$y^{\mathrm{I}} = a + x + dx$$
$$- \sqrt{xx + 2xdx + dx^2 + ax + adx - (x + dx)\sqrt{2ax - xx + 2adx - 2xdx - dx^2}}$$

But having put $x = a$ it will be

$$y^{\mathrm{I}} = 2a + dx - \sqrt{2aa + 3adx + dx^2 - (a + dx)\sqrt{aa - dx^2}}.$$

Now, because it is $\sqrt{aa - dx^2} = a - \frac{dx^2}{2a}$ (for, the following terms can surely be neglected, since not all which are infinitely smaller, will we cancelled, as it will become plain soon), it will be

$$y^{\mathrm{I}} = 2a + dx - \sqrt{aa + 2adx + \frac{3}{2}dx^2}$$

and further by extracting the root it will be

$$y^{\mathrm{I}} = 2a + dx - \left(a + dx + \frac{dx^2}{4a}\right) = a - \frac{dx^2}{4a}.$$

But in the case $x = a$ it will be $y = a$; hence, because it is $y^{\mathrm{I}} = y + dy$, one will obtain

$$dy = -\frac{dx^2}{4a};$$

from this it is at the same time seen that the propounded function y becomes imaginary, if one puts $x = a$.

EXAMPLE 2

To find the differential of this function $y = 2ax - xx + a\sqrt{aa - xx}$ in the case in which one puts $x = a$.

Having differentiated in the usual way it is

$$dy = 2adx - 2xdx - \frac{axdx}{\sqrt{aa - xx}},$$

which for $x = a$ goes over into infinity and hence is not indicated this way. But the differentials of the following orders in a similar way will become infinite such that from them not even from the series $pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^2 +$ etc. the true value of the differential can be found. Therefore, let us put $x + dx$ instead of x and we will have

$$y^{\mathrm{I}} = 2ax - xx + 2adx - 2xdx - dx^2 + a\sqrt{aa - xx - 2xdx - dx^2}$$

and having put $x = a$ it will be

$$y^{\mathrm{I}} = aa - dx^2 + a\sqrt{-2adx - dx^2}.$$

But in the same case it is $y = aa$; hence, it will be $dy = -dx^2 + a\sqrt{-2adx}$, and since dx^2 vanishes with respect to $\sqrt{-2adx}$, it will be

$$dy = a\sqrt{-2adx}.$$

Hence, if the differential dx is taken positively, dy will be imaginary; but if one writes $x - dx$ for x, it will be

$$dy = \sqrt{2adx};$$

because its value is a double value, the one positive, the other negative, the function y in the case y will be neither maximal nor minimal.

EXAMPLE 3

To find the differential of the function $y = 3aax - 3axx + x^3 + (a - x)\sqrt[3]{a^3 - x^3}$ in the case in which one puts $x = a$.

Since this function is transformed into this form

$$y = a^3 - (a - x)^3 + (a - x)^{\frac{7}{3}}\sqrt[3]{aa + ax + xx},$$

having put $x = a + dx$ it is

$$y^{\mathrm{I}} = a^3 + dx^3 - dx^{\frac{7}{3}}\sqrt[3]{3aa}$$

and in the same case it is $y = a^3$. Therefore, it will be $dy = dx^3 - dx^{\frac{7}{3}}\sqrt[3]{3aa}$, and because dx^3 vanishes with respect to to $dx^{\frac{7}{3}}$, it will be

$$dy = -dx^{\frac{7}{3}}\sqrt[3]{3aa};$$

therefore, in the case $x = a$ the function y becomes neither a maximum nor a minimum.

EXAMPLE 4

To find the differential of the function $y + \sqrt{x} + \sqrt[4]{x^3} = (1 + \sqrt[4]{x})\sqrt{x}$ *in the case* $x = 0$.

Since the case $x = 0$ is propounded and in it it is $y = 0$, instead of x only write dx and one will have

$$dy = dx^{\frac{1}{2}} + dx^{\frac{3}{2}} \quad \text{or} \quad dy = (1 + \sqrt[4]{dx})\sqrt{dx};$$

hence, at first it is plain that dx cannot be taken negatively. But then, even though $\sqrt{dx}$ has a double value, the one positive, the other negative, nevertheless in this case, since its root $\sqrt[4]{dx}$ occurs, only the positive value can be taken. But $\sqrt[4]{dx}$ has both meanings and it will be

$$dy = \sqrt{dx} \pm \sqrt[4]{dx^3} \quad \text{and} \quad y^{\mathrm{I}} = 0 + \sqrt{dx} \pm \sqrt[4]{dx^3}$$

because of $y = 0$. Because both values of y^{I} are greater than y, it follows that in the case $x = 0$ y become a minimum. But that the function $y = \sqrt{x} + \sqrt[4]{x^3}$ does not contain this one $-\sqrt{x} + \sqrt[4]{x^3}$, will become plain by making both rational. For, the first brought into this form $y - \sqrt{x} = \sqrt[4]{x}$ and squared gives $y^2 - 2y\sqrt{x} + x = x\sqrt{x}$ or $y^2 + x = (x + 2y)\sqrt{x}$ which squared again yields

$$y^4 - 2yyx - 4xxy + xx - x^3 = 0$$

The other $y + \sqrt{x} = \sqrt[4]{x^3}$ will give $y^2 + x = (x - 2y)\sqrt{x}$ and further

$$y^4 - 2yyx + 4xxy + xx - x^3 = 0,$$

which is different from the latter. But on the other hand, the term $\sqrt[4]{x^3}$ retains the ambiguity of the sign. Therefore, this circumstance is to be considered in detail, that, even though in general the roots of even powers include both signs $+$ and $-$, nevertheless this ambiguity does not occur, if in the same expression of the same roots higher roots of even powers occur; these would be imaginary, if the first roots would be taken negatively. And from this source maxima and minima of the second kind follow, whenever such might not seem to occur.

EXAMPLE 5

To fin the differential of the function

$$y = a + \sqrt{x-f} + (x-f)\sqrt[4]{x-f} + (x-f)^2\sqrt[8]{x-f}$$

in the case in which one puts $x = f$.

Let us put $x - f = t$, and because it is $y = a + \sqrt{t} + t\sqrt[4]{t} + tt\sqrt[8]{t}$, the differential of this is in question in the case $t = 0$ in which it is $y = a$. Therefore, having put $t + dt$ or $0 + dt$ instead of t it will be

$$y^{\mathrm{I}} = y + dy = a + \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}$$

and hence one will have

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}.$$

Here, at first it is plain that the differential cannot be taken negatively, otherwise dy becomes imaginary. But then not only $\sqrt{dt}$, but even $\sqrt[4]{dt}$ cannot be taken negatively; for, $\sqrt[8]{dt}$ would become imaginary; hence the differential dy has only the double value

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} \pm dt^2\sqrt[8]{dt};$$

because both values a greater then zero. it follows that the function y becomes a minimum of the second kind having put $t = 0$ or $x = f$. Although in these cases the terms $dt\sqrt[4]{dt}$ and $dt^2\sqrt[8]{dt}$ vanish with respect to the first $\sqrt{dt}$, it is nevertheless to be taken into account, if the multiplicity of the values is considered, that the imaginary quantities are avoided.

EXAMPLE 6

To find the differential of the function $y = ax + bxx + (x-f)^n + (x-f)^{m+\frac{1}{2}}$ in the case $x = f$.

If one puts $x = f$, it will be $y = af + bff$, and if instead of x one puts $x + dx$ or $f + dx$, the closest value will arise as

$$y^{\mathrm{I}} = af + bff + adx + 2bfdx + bdx^2 + dx^n + dx^{m+\frac{1}{2}n},$$

such that it is

$$dy = adx + 2bfdx + bdx^2 + dx^n + dx^m\sqrt{dx^n}.$$

Therefore, if n is not an even number, the differential dx cannot be taken negatively. But the last term $dx^m\sqrt{dx^n}$ has an ambiguous sign; hence, the value of y^{I} will be a double value, both greater than the one of y, if $a+2bf$ was a positive quantity and the exponents n and $m+\frac{1}{2}n$ were greater than the unity. Therefore, the value of the function y in the case $x=f$ will be minimal and this happens, whether n is an integer number or fractional, as long as only the numerator in this case also was not an even number in that case.

§351 But this method to deduce differentials from the principles themselves especially has use in the case of transcendental functions, since in certain cases the differential found in the usual way either vanishes or seem to grow to infinity. But here species of the infinite and the infinitely small of such a kind appear which are not found in algebraic functions. For, because, if i denotes an infinite number, $\ln i$ is also infinte, but nevertheless to the number i and even to any power i^n, no matter how small the exponent n is set, has an infinitely small ratio, the fraction $\frac{\ln i}{i^n}$ will be infinitely small and cannot be finite, before the exponent n become infinitely small. Therefore, $\ln i$ will be homogeneous to i^n, if the exponent n was infinitely small. Now let us put $i=\frac{1}{\omega}$ while ω is an infinitely small quantity; $-\ln\omega$ will be homogeneous to $\frac{1}{\omega^n}$, if the exponent n is infinitely small, and hence $-\frac{1}{\ln\omega}$ will be homogeneous to ω^n; and hence, $-\frac{1}{\ln dx}$ will be infinitely small compared to dx^n, while n is an infinitely small fraction. So, if it was $y=-\frac{1}{\ln x}$, the differential of y in the case $x=0$ will be $=-\frac{1}{\ln dx}=dx^n$ and hence dy will have an infinite ratio to dx and to any power of dx; and with respect to $-\frac{1}{\ln dx}$ completely all powers of dx vanish, no matter how small their exponents were.

§352 Further, we also saw, if a was a number greater than unity and i was infinite, that then a^i will be infinite of such a high degree that with respect to it not only i but also any power of i vanishes; and i^n does not become homogeneous to a^i before the exponent n was augmented to infinity. Now, let $i=\frac{i}{\omega}$ such that ω denotes the infinitely small; $a^{\frac{1}{\omega}}$ will be homogeneous to $\frac{1}{\omega^n}$ while n denotes an infinitely large number and hence $a^{\frac{-1}{\omega}}$ or $\frac{1}{a^{1:\omega}}$ will be infinitely large compared to ω^n. Hence, $\frac{1}{a^{1:dx}}$ will be infinitely small, but vanishes with respect to all powers of dx, because it is homogeneous to the power dx^n, while n is an infinitely large number. Hence, if the differential of $y=\frac{1}{a^{1:x}}$ is in question in the case $x=0$, since it is $y=0$, it will be $dy=\frac{1}{a^{1:dx}}$ and hence is infinitely smaller than each power of dx.

§353 But if a was a number smaller than unity, then, because $\frac{1}{a}$ becomes larger than unity, the question is reduced to the preceding case. If one has the expression $a^{\frac{1}{\omega}}$, it by putting $a = 1 : b$ will be transformed into $b^{-\frac{1}{\omega}}$ or $\frac{1}{b^{1:\omega}}$ which because of $b > 1$ will be homogeneous to ω^n while n denotes any infinitely large number. Therefore, having mentioned these things in advance, we will be able to resolve the following examples.

Example 1

To find the differential of the function $y = xx - \frac{1}{\ln x}$ in the case $x = 0$.

Since for $x = 0$ it is $y = 0$, if we put $x + dx$ or $0 + dx$ instead of x, it will be

$$y^{\mathrm{I}} = dy = dx^2 - \frac{1}{\ln dx}.$$

But because $-\frac{1}{\ln dx}$ is homogeneous to dx^n while n denotes an infinitely small number, with respect to it dx^2 will vanish and it will be

$$dy = -\frac{1}{\ln dx} = dx^n.$$

But because the logarithms of negative numbers are imaginary, dx cannot be taken negatively and therefore in the case $x = 0$ the function y will be a minimum, but a minimum extending neither to the first nor the second kind. It certainly does not extend to the first kind, since y has not preceding very close values, but is only smaller than the following values, if x is set greater than nothing. But it also does not extend to second kind, since the following value to which it is compared are not double values; and therefore a third kind of maxima and minima arises that only occurs in logarithmic and transcendental functions, but never occurs in algebraic functions; this will be treated in the following part on curves in more detail.

Example 2

To find the differential of the function $y = (a - x)^n - x^n(\ln a - \ln x)^n$ in the case in which it is $x = a$.

This differential, if n is not an integer number, can be found from the general formula

$$dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.};$$

for, it will be

$$pdx = -n(a-x)^{n-1}dx - nx^{n-1}dx(\ln a - \ln x)^n + nx^{n-1}(\ln a - \ln x)^{n-1}dx,$$

which value having put $x = a$ vanishes; for, even if $n = 1$, it will be

$$pdx = -dx + dx = 0.$$

Therefore, if we proceed further, it will be

$$\frac{1}{2}qdx^2 = \frac{n(n-1)}{1\cdot 2}(a-x)^{n-2}dx^2 - \frac{n(n-1)}{1\cdot 2}x^{n-1}dx^2(\ln a - \ln x)^n + \frac{n^2}{2}x^{n-2}dx^2(\ln a - \ln x)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}x^{n-2}dx^2(\ln a - \ln x)^{n-1} - \frac{n(n-1)}{1\cdot 2}dx^2(\ln a - \ln x)^{n-2}.$$

Hence, if it was $n = 1$, it will be $\frac{1}{2}qdx^2 = \frac{dx^2}{2a}$ for $x = a$. In similar manner, if $n = 2$, one would have to proceed up to the term $\frac{1}{6}rdx^3$ and so fourth. Therefore, one will more conveniently use the principles of differentiation, and because for $x = a$ it is $y = 0$, if we put $x + dx$ or $a + dx$ instead of x, it will be

$$y^{\mathrm{I}} = (-dx)^n - (a+dx)^n(\ln a - \ln(a+dx))^n = y + dy = dy$$

because of $y = 0$. But it is

$$\ln(a+dx) = \ln a + \frac{dx}{a} - \frac{dx^2}{2a^2} + \frac{dx^3}{3a^3} - \text{etc.},$$

whence it is

$$dy = (-dx)^n - \left(a^n + na^{n-1}dx + \frac{n(n-1)}{1\cdot 2}a^{n-2}dx^2 + \text{etc.}\right)\left(-\frac{dx}{a} + \frac{dx^2}{2a^2} - \frac{dx^3}{3a^3} + \text{etc.}\right)^n$$
$$= \frac{n}{2a}(-dx)^{n+1}.$$

Therefore, in the case $x = a$ the differential dy in question of the propounded formula will be as follows:

$$\begin{array}{ll|llll}
\text{if} & n=1 & dy & = & \dfrac{dx^2}{2a}, & \text{as we found before} \\
\text{if} & n=2 & dy & = - & \dfrac{2dx^3}{2a} & \\
\text{if} & n=3 & dy & = & \dfrac{3dx^4}{2a} & \\
\text{if} & n=4 & dy & = - & \dfrac{4dx^5}{2a} & \\
\text{etc.} & & \text{etc.} & & &
\end{array}$$

Therefore, if n was an odd number, the function in the case $x = a$ becomes a minimum, but if n is an even number, neither a maximum nor a minimum; the same holds, if n was a fraction having an odd denominator. But is n was a fraction having an even denominator, then dx has to be taken negatively, that we do not get to imaginary quantities; and because of the ambiguous meaning the function will also neither be a maximum nor a minimum.

EXAMPLE 3

To find the differential of the function $y = x^x$ in the case $x = \frac{1}{e}$ where e denotes the number whose hyperbolic logarithm is $= 1$.

Since in general it is $dy = x^x dx(\ln x + 1)$, this differential in the case $x = \frac{1}{e}$ or $\ln x = -1$ vanishes. Therefore, compare this differential to the general form $pdx + \frac{1}{2}qdx^2 +$ etc.; it will be

$$p = x^x(\ln x + 1) \quad \text{and} \quad q = x^x(\ln x + 1)^2 + x^{x-1}$$

and having put $\ln x = -1$ or $x = \frac{1}{e}$ it will be

$$q = \left(\frac{1}{e}\right)^{\frac{1-e}{e}} = e^{\frac{e-1}{e}}.$$

Hence, the differential in question will be

$$dy = \frac{1}{2}e^{(e-1):e}dx^2$$

and therefore the function $y = x^x$ will be a minimum in the case $x = \frac{1}{e}$.

EXAMPLE 4

To find the differential of this function $y = x^n + e^{-1:x}$ *in the case in which it is* $x = 0$.

Since for $x = 0$ it is $y = 0$, if one puts $x = 0 + dx$, it will be

$$y^{\mathrm{I}} = dy = dx^n + \frac{1}{e^{1:dx}}.$$

But we saw that $\frac{1}{e^{1:x}}$ is homogeneous to the infinite power of dx or to dx^∞ and hence will vanish with respect to dx^n such that it is

$$dy = dx^n.$$

§354 What happens in the first differentials in certain cases, that they do not arise by means of the usual rules of differentiation, also happens in differentials of the second and third and higher order in the cases in which in the complete differential form

$$d.y = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 + \text{etc.}$$

some of the quantities q, r, s etc. either vanish or become infinite. Because it is [§ 339]

$$dd.y = qdx^2 + rdx^3 + \frac{7}{12}sdx^4 + \text{etc.},$$

if in which case it is $q = 0$, then it will be $ddy = rdx^3$; but if in the same case also r vanishes, then from this series the second differential cannot be found at all, but one will have to go back to the principles of differentials; by putting $x + dx$ instead of x find the value y^{I} and by putting $x + 2dx$ instead of x find the value of y^{II} having done which the true value of the second differential will be

$$ddy = dy^{\mathrm{I}} - dy = y^{\mathrm{II}} - 2y^{\mathrm{I}} + y.$$

If in similar matter the question is raised about the third differential, then furthermore in y instead of x write $x + 3dx$ and having found the value y^{III} it will be

$$d^3y = y^{\mathrm{III}} - 2y^{\mathrm{II}} + 3y^{\mathrm{I}} - y$$

and so fourth. We will illustrate these cases in the following examples.

EXAMPLE 1

To find the second differential of the function $y = \frac{aa-xx}{aa+xx}$ in the case is which one puts $x = \frac{a}{\sqrt{3}}$.

By searching for the complete differential of y from the form

$$dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 + \text{etc.}$$

the following values will arise for p, q, r, s etc.

$$p = -\frac{4aax}{(aa+xx)^2}, \quad q = \frac{-4a^4 + 12aaxx}{(aa+xx)^3} \quad \text{and} \quad r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}.$$

Since now it is

$$ddy = rdx^3 + \frac{7}{12}sdx^4 + \text{etc.}$$

because of $q = 0$ in the case $x = \frac{a}{\sqrt{3}}$ and in the same case it is $r = \frac{27\sqrt{3}}{8a^3}$, the second differential in question will be

$$ddy = \frac{27dx^2\sqrt{3}}{8a^3}.$$

EXAMPLE 2

To find the third differential of the function $y = \frac{aa-xx}{aa+xx}$ in the case $x = a$.

As before by finding the complete differential

$$dy = \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \frac{1}{24}sdx^4 + \text{etc.},$$

since the third differential is $d^3y = rdx^3 + \frac{3}{2}sdx^4 + \text{etc.}$, because of

$$r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}$$

it will be $x = a$ in the case $r = 0$; hence, it is proceed to the value s which will be

$$s = \frac{48a^4 - 144aaxx}{(aa+xx)^4} - \frac{8x(48a^4x - 48aax^3)}{(aa+xx)^5};$$

therefore, having put $x0a$ it will be $s = -\frac{96a^4}{2^4a^8} = -\frac{6}{a^4}$; hence, it will be in this case

$$d^3y = -\frac{9dx^4}{a^4}.$$

EXAMPLE 3

To find the differential of arbitrary grade of the function $y = ax^m + bx^n$ in the case $x = 0$.

By successively putting $x + dx$, $x + 2dx$, $x + 3dx$ etc. instead of x the following values of the function y will be

$$\begin{aligned}
y^{\mathrm{I}} &= a(x+dx)^m \;\; + b(x+dx)^n, \\
y^{\mathrm{II}} &= a(x+2dx)^m + b(x+2dx)^n, \\
y^{\mathrm{III}} &= a(x+3dx)^m + b(x+3dx)^n, \\
&\text{etc.}
\end{aligned}$$

Therefore, having put $x = 0$ it will be $y = 0$ and its differentials will be

$$\begin{aligned}
dy &= adx^m + bdx^n, \\
ddy &= (2^m - 1)adx^m + (2^n - 2)bdx^n, \\
d^3y &= (3^m - 3 \cdot 2^m + 3)adx^m + (3^n - 3 \cdot 2^n + 3)bdx^n, \\
d^4y &= (4^m - 4 \cdot 3^m + 6 \cdot 2^m - 4)adx^m + (4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4)bdx^n \\
&\text{etc.}
\end{aligned}$$

Therefore, if the exponent n was greater than m, the second terms in these expressions vanish with respect to the first. Nevertheless it is to be considered, if n was a fractional number, that the cases in which these differentials become either imaginary or ambiguous can be distinguished. It will be convenient to keep up the further expansions of these cases for the doctrine of curves.